THE FIXED POINT PROPERTY OF A-DIRECT SUMS OF N UNIFORMLY NON-SQUARE BANACH SPACES (Nonlinear Analysis and Convex Analysis)

Tamura, Takayuki; Kato, Mikio

数理解析研究所講究録 第2114巻 2019年 133-136

THE FIXED POINT PROPERTY OF A-DIRECT SUMS OF N UNIFORMLY NON-SQUARE BANACH SPACES

TAKAYUKI TAMURA
Graduate School of Social Sciences, Chiba University
ttakayuki@faculty.chiba-u.jp and
MIKIO KATO
Kyushu Institute of Technology
katom@mns.kyutech.ac.jp

Abstract
We shall show the fixed point property of A-direct sums of N uniformly non-square Banach spaces by characterizing the nontrivialness of Dominguez-Benavides coefficient $R(1,(X_1\oplus\cdots\oplus X_N)_A)$, that is, $R(1,(X_1\oplus\cdots\oplus X_N)_A)<2$.

A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *monotone* if $\|\boldsymbol{a}\|\leq\|\boldsymbol{b}\|$ for all $\boldsymbol{a}=(a_j),\boldsymbol{b}=(b_j)\in\mathbb{R}^N$ with $|a_j|\leq|b_j|$ $(j=1,\ldots,N)$. For $\boldsymbol{a}=(a_j)$ $|\boldsymbol{a}|$ is defined by $|\boldsymbol{a}|=(|a_j|)\in\mathbb{R}^N$. A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *absolute* if $\|\boldsymbol{a}\|=\||\boldsymbol{a}|\|$ for all $\boldsymbol{a}\in\mathbb{R}^N$ and *normalized* if $\|e_j\|=1$ for all $1\leq j\leq N$, where e_j is the j-th unit vector in $\mathbb{R}^N$.

In [4] and [10] A-direct sums and AN-direct sums of N Banach spaces were introduced respectively by the following: Let $\|\cdot\|_A$ be an *arbitrary* norm on $\mathbb{R}^N$. The *A-direct sum* $(X_1\oplus\cdots\oplus X_N)_A$ is the direct sum of $X_1,\ldots,X_N$ equipped with the norm

$$\|(x_1,\ldots,x_N)\|_A=\|(\|x_1\|,\ldots,\|x_N\|)\|_A,\ (x_1,\ldots,x_N)\in X_1\oplus\cdots\oplus X_N$$

and an *AN-direct sum* is an A-direct sum whose norm is defined from some absolute noramlized norm $\|\cdot\|_{AN}$ on $\mathbb{R}^N$. It is known that a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ is aboslute if and only if it is monotone ([2],[4],[14]).

In [13] Z-direct sums were introduced by the following: Let $\|\cdot\|_Z$ be an *monotone* norm on $\mathbb{R}^N_+$. The *Z-direct sum* $(X_1\oplus\cdots\oplus X_N)_Z$ is the direct sum of $X_1,\ldots,X_N$ equipped with the norm

$$\|(x_1,\ldots,x_N)\|_Z=\|(\|x_1\|,\ldots,\|x_N\|)\|_Z,\ (x_1,\ldots,x_N)\in X_1\oplus\cdots\oplus X_N$$

Then we see that Z-direct sum and AN-direct sum are A-direct sum. Since an A-direct sum is isometric isomorphic to some AN-direct sum ([4]), then we have the following theorem.

Theorem 1 (cf. [4]). *Let $X_1,\ldots,X_N$ be Banach spaces. Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the norm of $(X_1\oplus\cdots\oplus X_N)_A$ is monotone, that is,*

$$\|(x_1,\ldots,x_N)\|_A\leq\|(y_1,\ldots,y_N)\|_A$$

for $(x_1, \dots, x_N), (y_1, \dots, y_N) \in (X_1 \oplus \cdots \oplus X_N)_A$ *with* $\|x_j\| \leq \|y_j\|$ $(j = 1, \dots, N)$.

As usual S_X and B_X stand for the unit sphere and the closed unit ball of X, respectively. A Banach space X is said to have the *fixed point property* (resp. *weak fixed point property*) for nonexpansive mappings if every nonexpansive self-mapping T of any nonempty bounded closed (resp. weakly compact) convex subset C of X has a fixed point (T is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). In [6] the coefficient $R(a, X)$ called as Dominguez-Benavides coefficient(cf. [3]) was introduced by the following: For $0 \leq a \leq 1$ let

$$R(a, X) = \sup \left\{ \liminf_{n \to \infty} \|x_n + x\| \right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences $\{x_n\}_n$ in the unit ball of X such that

$$\lim_{n,m \to \infty; n \neq m} \|x_n - x_m\| \leq 1.$$

In this paper we shall show the fixed point property for nonexpansive mappings of A-direct sums of N uniformly non-square Banach spaces by characterizing the nontrivialness of Dominguez-Benavides coefficient R(1, $(X_1 \oplus \cdots \oplus X_N)_A$), that is, R(1, $(X_1 \oplus \cdots \oplus X_N)_A) < 2$.

The following theorem was proved in [6].

Theorem 2 ([6]). *Let X be a Banach space. If $R(a, X) < 1 + a$ for some $a > 0$, then X has the weak fixed point property for nonexpansive mappings.*

A Banach space X is called *uniformly non-square* ([9]) if there exists a constant $\varepsilon > 0$ such that

$$\min\{\|x + y\|, \|x - y\|\} \leq 2(1 - \varepsilon) \quad \text{for all } x, y \in S_X.$$

By Theorem 2 García-Falset et. al. [8] obtained the following remarkable result.

Theorem 3 ([8]). *Let X be a uniformly non-square Banach space. Then $R(1, X) < 2$ and hence X has the fixed point property for nonexpansive mappings.*

In [7] the following notions were introduced.

Definition 4 ([7]). For $\boldsymbol{a} = (a_j) \in \mathbb{R}^N$ let supp $\boldsymbol{a} = \{j : a_j \neq 0\}$.

(i) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have *Property* T_1^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a} + \boldsymbol{b}\| = 1, \ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N \implies \text{supp } \boldsymbol{a} \cap \text{supp } \boldsymbol{b} \neq \emptyset.$$

(ii) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have *Property* T_∞^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{a} + \boldsymbol{b}\| = 1 \implies \text{supp } \boldsymbol{a} \cap \text{supp } \boldsymbol{b} \neq \emptyset.$$

To show our key result we need the following propositions.

Proposition 5 ([11]). *Let $\{x_n^{(k)}\}_{n,k}$, $\{y_n^{(k)}\}_{n,k}$ be double sequences with nonzero terms in a Banach space X such that*

$$\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\| > 0,\ \lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\| > 0.$$

Then the following are equivalent.

(i) $\lim_{k\to\infty}\liminf_{n\to\infty}\|x_n^{(k)} + y_n^{(k)}\| = \lim_{k\to\infty}\lim_{n\to\infty}(\|x_n^{(k)}\| + \|y_n^{(k)}\|)$.

(ii) $\lim_{k\to\infty}\liminf_{n\to\infty}\left\|\dfrac{x_n^{(k)}}{\|x_n^{(k)}\|} + \dfrac{y_n^{(k)}}{\|y_n^{(k)}\|}\right\| = 2.$

Proposition 6 ([5]; see also [1, Chpter III, Theorem 1.5]). *Let $\{x_n\}$ be a bounded sequence in a Banach space X. Then $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\lim_{k,l\to\infty;k\neq l}\|x_{n_k} - x_{n_l}\|$ exists.*

Proposition 7 ([16]). *Let $\{x_n\}$ be a weakly null sequence in a Banach space X. Assume that $\lim\limits_{n,m\to\infty;n\neq m}\|x_n - x_m\|$ exists. Then*

$$\limsup_{n\to\infty}\|x_n\| \le \lim_{n,m\to\infty;n\neq m}\|x_n - x_m\|.$$

Proposition 8 ([12]). *Let $\boldsymbol{a} = (a_j), \boldsymbol{b} = (b_j) \in \mathbb{R}^N$ and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ be monotone. If $\|\boldsymbol{a}\| = \|\boldsymbol{b}\|$, $|a_j| \le |b_j|$ $(j = 1, \dots, N)$ and $|a_{j_0}| < |b_{j_0}|$ then $\|(\chi_{\boldsymbol{N}\setminus\{j_0\}}(j)a_j)\| = \|(b_j)\|$, where $\boldsymbol{N} = \{1, \dots, N\}$ and $\chi_{N\setminus\{j_0\}}$ is the characteristic function of $\boldsymbol{N}\setminus\{j_0\}$.*

By Theorem 1, Propositions 5, 6, 7 and 8 we can prove the following key result.

Theorem 9. *Let $X_1, \dots, X_N$ be Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_1^N. Then $R(1, (X_1 \oplus \cdots \oplus X_N)_A) < 2$ if and only if $R(1, X_j) < 2$ for all $1 \le j \le N$*

Corollary 10 (cf. [15]). *Let X and Y be Banach spaces and let a norm on $\mathbb{R}^2$ be not ℓ_1-norm. Then $R(1, (X \oplus_A Y)) < 2$ if and only if $R(1, X) < 2$ and $R(1, Y) < 2$.*

Theorem 9 combined with Theorem 3 yields nontrivialness of Dominguez-Benavides coefficient of A-direct sums of N uniformly non-square Banach spaces.

Theorem 11. *Let $X_1, \dots, X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_1^N. Then $R(1, (X_1\oplus\cdots\oplus X_N)_A) < 2$.*

By Theorem 11 and Theorem 2 we obtain our main results.

Theorem 12. *Let $X_1, \dots, X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_1^N. Then $(X_1 \oplus \cdots \oplus X_N)_A$ has the fixed point property for nonexpansive mappings.*

Theorem 13. *Let $X_1, \ldots, X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_∞^N. Then $(X_1^* \oplus \cdots \oplus X_N^*)_{A^*}$ has the fixed point property for nonexpansive mappings, where $(X_1^* \oplus \cdots \oplus X_N^*)_{A^*}$ is an A-direct sum of $X_1^*, \ldots, X_N^*$ whose norm is defined by the dual norm $\|\cdot\|_A^*$.*

Acknowledgments

The authors were supported in part by JSPS KAKENHI (C) Grant number JP26400131.

References

[1] J.M. Ayerbe, T. Domínguez Benavides and G. Lopez, *Measure of noncompactness in metric fixed point theory*, Birkhaüzer, 1997

[2] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[3] Y. Cui, H. Hudzik, M. Wisła, *M-Constants, Dominguez-Benavides coefficient, and weak fixed point property in Orlicz sequence spaces equipped with the p-Amemiya norm*, Fixed Point Theory Appl. **2016:89**(2016), 1-14.

[4] S. Dhompongsa, M. Kato and T. Tamura, Uniform non-squareness for A-direct sums of Banach spaces with a strictly monotone norm, Linear Nonlinear Analysis **1** (2015), 247-260.

[5] T. Domínguez Benavides, *Some properties of the set and ball measures of noncompactness and applications*, J. London Math. Soc. **34** (1986), 120–128.

[6] T. Domínguez Benavides, *A geometrical coefficient implying the fixed point property and stability results*, Houston J. Math. **22** (1996), 835–849.

[7] P. N. Dowling and S. Saejung, Non-squareness and uniform non-squareness of Z-direct sums, J. Math. Anal. Appl. **369** (2010), 53–59.

[8] J. Garcia-Falset, E. Llorens-Fuster and Eva M. Mazcunan-Navarro, Uniformly non-square Banach spaces have the fixed point property for nonexpansive mappings, J. Funct. Anal. **233** (2006), 494-514.

[9] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. **80** (1964), 542–550.

[10] M. Kato, K.-S. Saito and T. Tamura, On the ψ-direct sums of Banach spaces and convexity, J. Aust. Math. Soc. **75** (2003), 413–422.

[11] M. Kato, K.-S. Saito and T. Tamura, Uniform non-ℓ_1^n-ness of ψ-direct sums of Banach spaces, J. Nonlinear Convex Anal. **11** (2010), 13–33.

[12] M. Kato and T. Tamura, Weak nearly uniform smoothness of the ψ-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$, Comment. Math. **52** (2012), 171-198.

[13] T. R. Landes, Permanence properties of normal structure, Pacific J. Math. **10** (1984), 125-143.

[14] K.-S. Saito, M. Kato and Y. Takahashi, On absolute norms on $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879–905.

[15] T. Tamura, $M(X \oplus_\psi Y)$ for the ψ-direct sum of two Banach spaces X and Y, Studies on Humanities and Social Sciences of Chiba University **25**(2012), 1-9.

[16] T. Tamura, On Dominguez-Benavides coefficient of ψ-direct sums $(X_1 \oplus \oplus X_N)_\psi$ of Banach spaces, Linear and Nonlinear Analysis **3** (2017), 87-99.